used in coupling structures for solid-state oscillators and in waveguide filter and impedance matching applications. The noncontacting movable-susceptance element is easy to realize and should find application in numerous laboratory devices.

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REFERENCES

[1] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, *Principles of Microwave Circuits* (M.I.T. Radiation Lab. Series), vol. 8. Boston, Mass.: Boston Technical Publishers, 1964, pp. 162–167.
[2] N. Marcuvitz, *The Waveguide Handbook* (M.I.T. Radiation Lab. Series), vol. 10. Boston, Mass. Boston Technical Publishers, 1964, pp. 218–221.

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A Class of Equiripple Functions which Complement the Achieser (or Zolotarev) Polynomials

***Abstract*—A symmetrical cascade of N commensurate transmission lines having equal ripple performance over a passband centered at the quarter-wavelength frequency may be synthesized using Achieser (or Zolotarev) polynomials for N odd. This correspondence identifies the solution of the case where N is even, and a new class of functions which complement these polynomials is identified.**

Levy [1] has made available for engineering use a class of odd polynomials, discovered by Achieser [2] which are equiripple in two symmetrical line segments $[-1, -\lambda]$ and $[\lambda, 1]$. These polynomials, however, do not completely solve the approximation problem for equiripple performance from a symmetrical cascade of equal-length transmission line elements (TLEs), all with the same propagation constant and electrical length θ.

As the author [3] has observed, the insertion loss function, P_L, of such a cascade may be written

$$P_L = 1 + [\sin\theta Q_{N-1}(\cos\theta)]^2 = 1 + P_N{}^2(\sin\theta, \cos\theta) \qquad (1)$$

where N is the number TLEs. More specifically, $Q_{N-1}(\cos\theta)$ is an even function of $\cos\theta$ for N odd and an odd function of $\cos\theta$ for N even. For N odd, putting $x = \sin\theta$ and $x^2 = 1 - \cos^2\theta$,

$$\sin\theta Q_{N-1}(\cos\theta) = P_N(\sin\theta) \qquad (2)$$

where $P_N(x)$ is an odd polynomial. When $\sin\theta = 1$ at midband, it is readily seen that the Achieser polynomials will give equiripple performance over a given band centered at the quarter-wave frequency with band edges at $\theta = \sin^{-1/2}\lambda$ and $\theta = \pi - \sin^{-1}\lambda$. The polynomials of degree $N = 2n$, which are equiripple in the same intervals are

$$[(T_n 2x^2 - 1 - \lambda^2)/(1 - \lambda^2)] \qquad (3)$$

but the $P_{2n}(\sin\theta)$ which is appropriate to the bandpass case cannot be expressed in this form.[1]

It is the object of this correspondence to identify the missing functions and so solve the approximation problem for equal-ripple behavior for cascades of the type under consideration. For N even, we require a function

$$P_{2n}(x) = x\sqrt{1 - x^2}\, Q_{n-1}{}^{(x^2)} \qquad (4)$$

which is equiripple over the given intervals of x, where Q_{n-1} is a polynomial of degree $n-1$ with real coefficients. Then with $x = \sin\theta$,

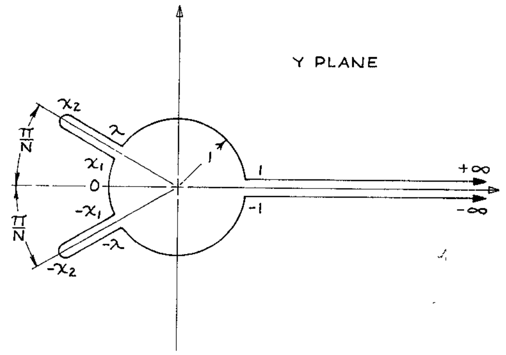

Fig. 1. Map of real axis of x plane.

the function $P_{2n}(x)$ will give equiripple performance over a band centered at $\theta = \pi/2$ and the synthesis will result in a symmetrical cascade having an even number of TLEs.

The functions (4) are defined parametrically by

$$P_N(x) = \frac{1}{2j}\left[\left(\frac{H(M+u)}{H(M-u)}\right)^n - \left(\frac{H(M-u)}{H(M+u)}\right)^n\right] \qquad (5)$$

$$x = \frac{\lambda \operatorname{cn}(u)}{\sqrt{\lambda^2 - \operatorname{sn}^2(u)}} \qquad (6)$$

where $\lambda = \operatorname{sn} M$, K is the complete elliptic integral of the first kind, and all the periodic functions have a common modulus, k, chosen to satisfy the equation, $NM = K$.

The elliptic function in (5) is clearly of the type used by Zolotarev [5] in his problem. It has however "sine like" form rather than the familiar "cosine like" equiripple form. For this reason, and because no other justification is given for its correctness, the analytic features of the following proof are more detailed than those given by Achieser [2] and Levy [1] while, in the interest of brevity, the formal features are left to the reader.

In the first place, (6) maps the interior of the rectangle, on a properly defined Riemann surface, u, bounded by $(0 \pm jK')$ and $(K \pm jK')$ into the upper half of the x plane while the transformation

$$y = \frac{H\left(\frac{K}{N} + u\right)}{H\left(\frac{K}{N} - u\right)} \qquad (7)$$

maps the same rectangle in the u plane into the exterior of the unit circle in the y plane. Fig. 1 establishes the location of points in the y plane which correspond to the points of greatest interest in the x plane, as shown in [1, Fig. 3].

For $|x|$ between λ and 1, (5) can be written

$$P_N(x) = \operatorname{Im}[y^n]. \qquad (8)$$

Thus as $|x|$ decreases from 1 to λ, $P_N(x)$ oscillates $n-1$ times between ± 1, taking on the value at λ, $\operatorname{Im}[e^{j\pi(n-\frac{1}{2})}] = \pm 1$, depending on whether n is even or odd; so that $P_N(x)$ has the required equiripple performance in the bands of interest.

It remains to show that it has the required form and it will be convenient to write,

$$x = \frac{\operatorname{sn}(M)\operatorname{cn}(u)}{\sqrt{\operatorname{sn}^2(M) - \operatorname{sn}^2(u)}} \qquad (9)$$

and

$$\sqrt{1 - x^2} = \frac{j\operatorname{cn}(M)\operatorname{sn}(u)}{\sqrt{\operatorname{sn}^2(M) - \operatorname{sn}^2(u)}}. \qquad (10)$$

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[1] For a problem in which these classes of polynomials do complement each other see [4].

Now, using a well-known addition theorem for the theta functions,

$$x\sqrt{1-x^2} = j\frac{K'H(M)H_1(M)H(u)H_1(u)}{K\theta^2(0)H(M+u)H(M-u)} \tag{11}$$

so that

$$P_N(x)/(x\sqrt{1-x^2}) = C\frac{H^N(M+u) - H^N(M-u)}{H(u)H_1(u)H^{n-1}(M+u)H^{n-1}(M-u)} \tag{12}$$

where C is a real constant. Denoting the fraction on the right by $F(u)$, the argument will be complete when we show that the function, $f(x)$, defined by $F(u)$ together with (6), is an even polynomial in x with real coefficients.

That $f(x)$ is a polynomial of degree $N-2$ is evident from the following.

1) $f(x)$ is single valued.

2) $f(x)$ is analytic everywhere except at the point of infinity.

3) The singularity of $f(x)$ at infinity is a pole of order $N-2$, for it is well known that an analytic function whose only singularity is a pole at infinity of order m is a polynomial of degree m. Then, to show that it is an even polynomial with real coefficients, it is sufficient to show that it is even and real on the portion of the real axis between $-x_1$ and $+x_1$, by the principle of analytic continuation.

Now $f(x)$ is single valued because $F(u)$ is doubly periodic in u with the same periodicity rectangle that x has as a function of u. So that, although to each value of x there corresponds an infinite number of values of u, each in turn gives the same value of $F(u)$. To show that $F(u)$ has the required property is a formal matter; and the reader is referred to [1]. One simply replaces u by $u+2K$ and $u+2jK'$, in turn, in the defining equations and shows that

$$F(u) = F(u+2K) = F(u+2jK') \tag{13}$$

making use of the fact that $NM=K$.

By the function of a function theorem for analytic functions, $f(x)$ is analytic except at the singularities of $F(u)$ as a function of u and the singularities of u as function of x. Both of these sets of singularities are readily seen to be finite in number. It follows then, from a well-known theorem for single-valued analytic functions with a finite number of singularities, that $f(x)$ cannot be bounded in the neighborhood of any of its singularities. Hence, in searching for the singularities of $f(x)$, we need not concern ourselves with the critical points of (6) since $f(x)$ can be unbounded only when $F(u)$ is unbounded. Moreover, because of the periodicity of $F(u)$, we may limit the search for singular points to values of u in a periodicity rectangle determined by $(\pm K, \pm jK')$. Now, since $H(u)$ is bounded in the finite plane, infinities of $F(u)$ occur only at zeros of its denominator. The zeros of $H(u)$ and $H_1(u)$ are simple and occur at $u=0$ and $u=-K$, but it is readily seen that they are cancelled by zeros of the numerator of $F(u)$. Thus the only singularities of $F(u)$ in the periodicity rectangle occur at $u=\pm M$. Both of these values correspond to $x=\infty$ and we conclude that $f(x)$ is analytic in the entire plane except at the point of infinity.

It is a formal matter to show that $f(x)/x^{N-2}$ approaches a finite limit as $x\rightarrow\infty$. Thus the only singularity of $f(x)$ is a pole of order $N-2$ at infinity.

Finally, from the mapping of the x plane (given in Fig. 1) and the evaluation of $P_N(x)$ as $\mathrm{Im}[y^n]$ for values of x between $-x_1$ and $+x_1$, it follows the $P_N(x)$ is a real odd function of x on this line segment. Thus $f(x)$ is a real even polynomial for values of x on this line segment.

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REFERENCES

[1] R. Levy, "Generalized rational function approximation in finite intervals using Zolotarev functions," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-8, Dec. 1970, pp. 1052–1064.
[2] N. I. Achieser, "Uber einige Funktionen, die in gegebenen intervallen am wenigsten von null abweichen," *Bull Soc. Phys. Math. de Kazan*, vol. 3, p. 111.
[3] H. J. Riblet, "General synthesis of quarter-wave impedance transformers," *IRE Trans. Microwave Theory Tech.*, vol. MTT-5, Jan. 1957, pp. 36–43.
[4] C. L. Dolph and H. J. Riblet, "Discussion on 'A current distribution for broadside arrays which optimizes the relationship between beam width and side-lobe level,' " *Proc. IRE*, vol. 35, May 1947, pp. 489–492.
[5] E. Zolotarev, "The application of elliptic functions to questions concerning functions which most or least deviate from zero," (in Russian), *Notes Russian Sci. Acad.*, vol. 30, 1877, appendix 5, pp. 1–71.

Computer Program Descriptions

Computer Solution of Transient and Time Domain Thin-Wire Antenna Problems

PURPOSE: SWIRE is a general purpose computer program which analyzes the transient and time domain electromagnetic behavior of straight-wire scatterers and antennas (both transmitting and receiving).

LANGUAGE: FORTRAN.

AUTHORS: Alethia M. Auckenthaler and C. Leonard Bennett, Sperry Rand Research Center, Sudbury, Mass. 01776.

AVAILABILITY: ASIS-NAPS Document No. NAPS-01541.

DESCRIPTION: A general purpose computer program that analyzes the transient and time domain electromagnetic behavior of transmitting, receiving, and scattering straight-wire antennas is presented. The program allows an arbitrary number of transmit or receive points, each with arbitrary source or load resistances and arbitrary distributed resistive loading along the wire length. The program also permits the computation of the far zone normalized field in arbitrary directions. The flexibility in both the input and output of this program and its applicability to the general time varying case allows the solution of a wide range of practical engineering problems.

The straight-wire scattering and antenna problem which is illustrated in Fig. 1 consists of a straight wire located on the x axis with some excitation. For the case of the scattering or receiving antenna problem the excitation is the x component of the incident wave E^i which makes an angle of θ^i with the plane perpendicular to the x axis. For the case of the transmitting antenna problem the excitation is a voltage generator with a source resistance R_g. These excitations produce currents $I(x)$ along the wire which in turn produce a far zone field H^s in the Ψ^s direction.

The technique used to solve this wire scattering problem [1], [2] is a specialization of the integral equation technique used in the time domain solution of the more general problem of scattering by surfaces [3]. Since the wire is assumed to be thin (e.g., the wire radius is much less than the width of an incident Gaussian shaped pulse), the wire current flows only in the axial direction and the more complicated surface integral equation reduces to a single space time scalar

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